TRANSIENT THERMAL CONVECTION IN A SPHERICAL CAVITY

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Comparatively little has been published about transient thermal convection in partly open regions, for instance in Ref. [1,2,3,4] only one, namely Ref. [4] deals with the case of a spherical film. In this article we give an approximate solution to one problem concerning slowly varying thermal convection inside a spherical cavity. This particular solution differs from those normally employed previously Ref. [1,3,4].

1. Basic Problem. 1. Assume that at the initial instant $t = 0$ liquid is at rest at temperature T_0 and fills a spherical cavity of radius R, the walls of which are kept permanently at a temperature T_1, which differs from the initial temperature of the liquid, i.e. let us say that $T_0 > T_1$. This kind of problem might represent, rather roughly, the case of liquid cooling in a spherical container, the walls of which are cooled by a stream outside.

2. We take the origin of a Cartesian coordinate system X, Y, Z at the centre of the sphere and the Z axis vertically upwards. We introduce the following nondimensional coordinates and time

$$\mathrm{X} = \frac{X}{R}, \qquad \mathrm{Y} = \frac{Y}{R}, \qquad \mathrm{Z} = \frac{Z}{R}, \qquad \mathrm{t} = \frac{\nu}{R^2} t^* \tag{1.1}$$

where ν is the kinematic viscosity. The quantities which appear in the convection equations (see Ref. [3]) are: hydrodynamic velocity $\mathbf{V}^*$, the pressure p^* in excess of the equilibrium pressure at temperature T_1, and the excess temperature Θ^* above T_1. We replace these by the corresponding nondimensional variables

$$\mathrm{V}^* = \frac{\nu}{R}\mathrm{V}, \qquad p^* = \rho_1 \left(\frac{\nu}{R}\right)^2 p, \qquad \theta^* = (T_0 - T_1)\,\theta^* \tag{1.2}$$

where ρ_1 is the equilibrium density at T_1.

The convection equations in Ref. [3] then become:

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\nabla)\mathbf{V} = -\nabla p + G\theta\mathbf{n} + \Delta\mathbf{V} \qquad \left(G = \frac{g\beta(T_0 - T_1)R^3}{\nu^2}\right)$$
$$\frac{\partial\theta}{\partial t} + (\mathbf{V}\Delta)\theta = \frac{1}{\sigma}\Delta\theta, \qquad \operatorname{div}\mathbf{V} = 0 \qquad \left(\sigma = \frac{\nu}{\chi}\right) \tag{1.3}$$

here χ is the thermal conductivity, $\mathbf{n}$ is the unit vector along the Z axis.

3. The initial and boundary conditions satisfying the requirements of paragraph 1 and the usual assumption of no slip at the wall are

$$\mathbf{V} = 0, \quad \theta = 1 \quad \text{for } t = 0; \qquad \mathbf{V} = 0, \quad \theta = 0 \quad \text{for } r = 1 \tag{1.4}$$

where r is the dimensionless radius vector.

In addition we require the solution to be finite in accordance with physical conditions at the centre of the sphere.

2. Method of Solution. Under the assumption that conditions vary slowly with time the corresponding Grashof number is small, and we may look for a solution of (1.3) with boundary conditions (1.4) in terms of a series for **V**, p and Θ in powers of Grashof number. Ref. [5].

$$\mathbf{V} = G\mathbf{V}^{(1)} + G^2\mathbf{V}^{(2)} + \ldots, \quad p = G(p^{(1)} + G^2p^{(2)} + \ldots \quad \theta = \theta^{(0)} + G\theta^{(1)} + G^2\theta^{(2)} + \ldots \tag{2.1}$$

Substituting these into (1.3) and into boundary condition (1.4), we get a linear equation for successive approximations. We will confine ourselves to zero and first approximations.

2. In the zero order approximation we get the equation

$$\frac{\partial\theta^{(0)}}{\partial t} = \frac{1}{\sigma}\Delta\theta^{(0)} \tag{2.2}$$

The boundary condition for this equation comes from (1.4), and the initial condition can be obtained from (1.4) so that when t = 0, $\Theta^{(0)}$ = 1. Thus the boundary conditions for (2.2) are,

$$\theta^{(0)} = 1 \quad \text{for } t = 0; \qquad \theta^{(0)} = 0 \quad \text{for } r = 1 \tag{2.3}$$

In the first order approximation we get the equations

$$\frac{\partial\mathbf{V}^{(1)}}{\partial t} = -\nabla P^{(1)} + \theta^{(0)}\mathbf{n} + \Delta\mathbf{V}^{(1)}, \quad \operatorname{div}\mathbf{V}^{(1)} = 0$$
$$\frac{\partial\theta^{(1)}}{\partial t} + (\mathbf{V}^{(1)}\Delta)\theta^{(0)} = \frac{1}{\sigma}\Delta\theta^{(1)}, \tag{2.4}$$

with zero boundary conditions

$$\mathbf{V}^{(1)} = 0, \quad \theta^{(1)} = 0 \quad \text{for } t = 0; \qquad \mathbf{V}^{(1)} = 0, \quad \theta^{(1)} = 0 \text{ for } r = 1 \tag{2.5}$$

and the condition that the solution is finite at the centre of the sphere.

3. The solution of the zero order approximation (2.2) with the boundary conditions (2.3) is known (see, for instance, Ref. [6]). To solve equations (2.4) in the first approximation with the corresponding boundary conditions we shall use the following method.

Introducing the notation div $\mathbf{V}^{(1)} \equiv \Omega$ and taking the divergence of the first equation (2.4) we obtain

$$\frac{\partial}{\partial t}\Omega - \Delta\Omega = -\Delta p^{(1)} + \operatorname{div}(\theta^{(0)}\mathbf{n}) \tag{2.6}$$

It follows from this that $p^{(1)}$ satisfies

$$\Delta p^{(1)} = \operatorname{div}(\theta^{(0)}\mathbf{n}) \tag{2.7}$$

since the second of equations (2.4) gives

$$\frac{\partial\Omega}{\partial t} - \Delta\Omega = 0 \tag{2.8}$$

Therefore $p^{(1)}$ must be found in the solution of (2.7). This equation is solved by a known method. The solutions, which are bounded at the centre of the sphere, contain unknown functions of time. We choose one of these solutions $p^{(1)}$ and the unknown functions of time so that, when $p^{(1)}$ is substituted into the first of equations (2.4) the resulting solution for $\mathbf{v}^{(1)}$ which satisfies the boundary conditions (2.5) and is bounded at the centre of the sphere leads to the conditions

$$\Omega = 0 \quad \text{for } r = 1 \tag{2.9}$$

Then Ω will satisfy equation (2.8), with the boundary condition (2.9) and the initial condition

$$\Omega = 0 \quad \text{for } t = 0 \tag{2.10}$$

which is valid because of (2.5). It follows that $\Omega = 0$, i.e. the second equation (2.4) is fulfilled, as is known from Ref. [7]. We can now solve the third of equations (2.4).

3. Solution of the problem and brief discussion of results. 1. When we substitute the solution to (2.2) with boundary conditions (2.3) (see, for instance Ref. [6])

$$\theta^{(0)} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2}{k\pi} \frac{\sin k\pi r}{r} \exp -\frac{k^2\pi^2 t}{\sigma} \tag{3.1}$$

into equation (2.7) we find that the solution for $p^{(1)}$ which is finite at the centre of the sphere is

$$p^{(1)} = -\left[\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2}{(k\pi)^3} \exp -\frac{k^2\pi^2 t}{\sigma} \frac{d}{dr} \frac{\sin k\pi r}{r} + s(t) r\right] \cos\psi \tag{3.2}$$

where ψ is the polar coordinate in the spherical system and $s(t)$ is, so far, an unknown function of time.

2. To solve the equations for the first approximation we use the complete system of orthogonal functions which are solutions to the equation

$$\Delta\Phi(r) + \lambda\Phi(r) = 0 \tag{3.3}$$

where λ is a constant, with the boundary condition $\Phi = 0$ for $r = 1$ and the condition that the solution is bounded at the centre of the sphere. This system of functions is as follows [7]:

$$r^{-\frac{1}{2}} J_{n+{}^1\!/_2}(\varepsilon_i^{(n)} r)\, P_n^{(m)}(\mu) \begin{matrix}\sin m\varphi \\ \cos m\varphi\end{matrix} \qquad (\mu \equiv \cos\psi) \tag{3.4}$$

where indices n, i, m have integral values. Here $J_{n+1/2}$ are Bessel functions, $\epsilon_i^{(n)}$ are their roots, $P_n^{(m)}$ associated Legendre functions, and ϕ is the azimuth coordinate in the spherical system.

If we substitute (3.1) and (3.2) into the first of equations (2.4), we obtain, by known methods, using (3.4)

$$V_x^{(1)} = \sum_{i=1}^{\infty} F_i(t)\, r^{-{}^1\!/_2} J_{2+{}^1\!/_2}(\varepsilon_i^{(2)} r)\, P_2^{(1)}(\mu) \cos\varphi \tag{3.5}$$

$$V_y^{(1)} = \sum_{i=1}^{\infty} F_i(t) r^{-{}^1\!/_2} J_{2+{}^1\!/_2}(\varepsilon_i^{(2)} r)\, P_{(2)}^{(1)} \mu \sin\varphi$$

$$V_z^{(1)} = 2\sum_{i=1}^{\infty} F_i(t) r^{-{}^1\!/_2} J_{2+{}^1\!/_2}(\varepsilon_i^{(2)} r)\, P_2(\mu) + \sum_{i=1}^{\infty} K_i(t)\, r^{-{}^1\!/_2} J_{{}^1\!/_2}(\varepsilon_i^{(0)} r)$$

where

$$F_i \equiv -\exp\left[-(\varepsilon_i^{(2)})^2 t\right] \int_0^t G_i(t) \exp(\varepsilon_i^{(2)})^2\, dt,$$

$$G_i = \frac{2}{[J'_{2+{}^1\!/_2}(\varepsilon_i^{(2)})]^2} \int_0^1 r^{{}^3\!/_2} H(r,t)\, J_{2+{}^1\!/_2}(\varepsilon_i^{(2)} r)\, dr, \qquad H \equiv \frac{1}{3}\left(\frac{\partial}{\partial r} h - \frac{1}{r} h\right),$$

$$h \equiv -\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2}{(k\pi)^3} \frac{d}{dr}\left(\frac{\sin k\pi r}{r}\right) \exp\frac{-k^2\pi^2 t}{\sigma} \tag{3.6}$$

and the coefficients $K_i(t)$ and the function $s(t)$ are determined by the system

$$\frac{d}{dt} K_i + (\varepsilon_i^{(0)})^2 K_i = N_i s + M_i, \qquad \sum_{i=1}^{\infty} \alpha_i K_i = -W \tag{3.7}$$

for zero initial conditions (the second of equations (3.7) is obtained from (2.9)). Moreover

$$M_i \equiv \frac{4}{3\vartheta} \int_0^1 r^{{}^3\!/_2} \theta^{(0)}(r,t)\, J_{{}^1\!/_2}(\varepsilon_i^{(0)} r)\, dr, \qquad N_i \equiv \frac{2}{\vartheta} \int_0^1 r^{{}^3\!/_2} J_{{}^1\!/_2}(\varepsilon_i^{(0)} r)\, dr$$

$$\vartheta = (J'_{{}^1\!/_2}(\varepsilon_i^{(0)}))^2 \tag{3.8}$$

$$\alpha_i \equiv -\varepsilon_i^{(0)} J_{1+{}^1\!/_2}(\varepsilon_i^{(0)}), \qquad \delta_i \equiv -2\varepsilon_i^{(2)} J_{3+{}^1\!/_2}(\varepsilon_i^{(2)}), \qquad W \equiv \sum_i \delta_i F_i \tag{3.9}$$

We obtain the solution of the third equation (2.4) in a similar way,

$$\theta^{(1)} = \sum_{j=1}^{\infty} B_j(t)\, r^{-1/2} J_{1+1/2}(\varepsilon_j^{(1)} r) \cos\psi \tag{3.10}$$

where

$$B_j \equiv -\exp\left[-(\varepsilon_j^{(1)})^2 t\right] \int_0^t Q_j(t) \exp\left[(\varepsilon_j^{(1)})^2 t\right] dt$$

$$Q_j \equiv \frac{2}{[J'_{1+1/2}(\varepsilon_j^{(1)})]^2} \int_0^1 r^{\frac{3}{2}} (2F(r,t) + K(r,t))\, J_{1+1/2}(\varepsilon_j^{(1)} r)\, dr \tag{3.11}$$

$$F \equiv \sum_{i=1}^{\infty} F_i r^{-1/2} J_{2+1/2}(\varepsilon_i^{(2)} r), \qquad K \equiv \sum_{i=1}^{\infty} K_i r^{-1/2} J_{1/2}(\varepsilon_i^{(0)} r)$$

The solution to the system of equations (3.7) for the coefficients K_i and the function $s(t)$, was found when only the first three terms were retained ($i = 1,2,3$). However we shall not introduce this here since it is unwieldy.

3. We note some special properties of convection, which can be deduced from the approximate solution to our problem.

Writing down the equation of flow in spherical coordinates, we have

$$\varphi = c_1, \quad \sin\psi = c_2 e^{f(r,t)} \quad \begin{pmatrix} c_1 = \text{const} \\ c_2 = \text{const} \end{pmatrix}, \qquad f \equiv \int \frac{u_1}{u_2}\, dr, \quad \begin{matrix} u_1 \equiv F - K \\ u_2 \equiv 2F + K \end{matrix} \tag{3.12}$$

It follows from (3.12) that the liquid moves in vertical (r, ψ) planes which go through the centre of the sphere, and the motion is of the same form in all these planes. The motion is axially symmetrical, the hotter liquid rising at the centre of the sphere. The liquid sinks, during cooling, at the walls. (If $T_0 < T_1$ the condition would be reversed.) Fig. 1 shows streamlines calculated from formulas (3.5) with the following values of parameters $\sigma = 6.75$ (water at 20°C,) $G = 300$, and the nondimensional time $t = 1$. Three coefficients K_i were used.

The temperature distribution in the liquid corresponds to its motion. Fig. 2, plotted with the same values as above, shows a plane section ϕ = const. revealing a family of isotherms at a spacing of 0.1. Formulas (3.1) and (3.10) were used in the calculations, with three terms in the summation (3.1) and three coefficients K_i. The temperature decreases in the direction away from the centre. The upper part of the region is, on the average, at a higher temperature than the lower. The liquid is cooled least in a region located at about one third radius above the centre (where there is purely molecular conductivity).

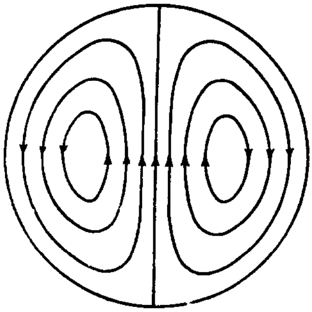

Fig. 1.

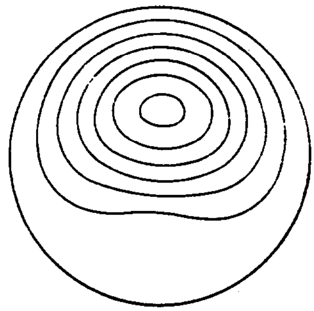

Fig. 2.

In order to assess the effect of convection, in this approximation, on the rate of cooling of the liquid as a whole, we consider the integral from the convective part of the temperature Θ over the volume V

$$\int \theta dV = \int \theta^{(0)} dV + G \int \theta^{(1)} dV + \cdots \tag{3.13}$$

For relatively small values of Θ (weak convection) this can be considered proportional, with sufficient accuracy, to the excess of the internal energy of the liquid over the value corresponding to full cooling of the liquid down to wall temperature T_1. If we insert the expression $\Theta^{(1)}$ from (3.10) we find that the second integral on the right vanishes. This shows that, in this approximation, the energy of the liquid is the same as that of a solid with the same thermal characteristics and with the same boundary conditions (the kinetic energy of the liquid is neglected since it is of the second order of magnitude compared with G). Thus the approximation shows that convection fails to affect the rate of cooling of the liquid as a whole. This approximation represents a temperature redistribution within the liquid such that the liquid in the top part of the space heats up at the expense of that in the lower. The effect of increasing the rate of cooling as a whole on convection should become clear in further approximations.

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